Behavior of Latent Vector of Trivariate Wishart Matrix

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abstract

This paper is concerned with the probability density function of the latent vector corresponding to the largest latent root of Wishart matrix. The latent vector may be expressed by the polar coordinates. Sugiyama (1966) give the exact expression of the probability density function of the polar coordinates. The function contained the alternating series, thus the function may not be converged on the domain of definition, numerically. In this paper we derived an improved expression of the function to be the positive series, for which we provide graphs of a population latent vector and latent roots.

1 Introduction

Let $x_1, \ldots, x_N$ be a random sample from the $p$-variate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$, which is denoted by $N_p(\mu, \Sigma)$. Then, the unbiased estimator of $\Sigma$ is given by

$$S = \frac{1}{N-1} \sum_{\alpha=1}^{N} (x_{\alpha} - \bar{x})(x_{\alpha} - \bar{x})'$$

where $\bar{x} = (1/n) \sum_{\alpha=1}^{N} x_{\alpha}$. In this case, $nS = (N-1)\tilde{S}$ is distributed according to the Wishart distribution with $n$ degrees of freedom and the covariance matrix $\Sigma$, denoted by $W_p(n, \Sigma)$. The latent roots and vectors of $\tilde{S}$ have important roles for statistical inference. For example, the principal components analysis results in the linear combination of latent vectors of $\Sigma$ and of the original variable vector. The variances of the principal components are given by the latent roots of $\Sigma$. Generally these parameters are unknown, thus we must estimate them using the latent roots and vectors of $\tilde{S}$. The statistical inferences containing the asymptotic distribution have been studied intensely. However, few papers have been published on latent vectors and how they may be expressed by their polar coordinates. Sugiyama (1965) gave the probability density function of the polar coordinates when $p = 2$. Later, Sugiyama (1966) extended this initial result to the general $p$. The main aim of this paper is to examine the stability of variation from the graph when $p = 3$. According to Sugiyama (1966), the probability density function is expressed by the alternating series, thus the
function may not be converged on the domain of definition numerically. Initially in this paper, we shall derive an expression by a positive term series, which is useful for numerical computation. We then drew the graphs of the probability density function when \( p = 3 \) and \( n = 4, 10 \) and 50. These graphs are unimodal on the region of definition.

The related works of this paper are as follows: Anderson (1963) gave an asymptotic distribution of the latent vectors of a Wishart matrix. Sugiura (1976) gave an asymptotic expansion of the distribution of the latent vector corresponding to the simple root of \( \Sigma \). Khatri and Pillai (1969) derived an exact distribution of the latent vectors corresponding to the largest latent roots in one and two sample cases. Takemura and Sheena (2007) derived an asymptotic normality of the latent vectors for the normalized sample latent roots when the population eigenvalues were infinitely dispersed.

This paper is organized as follows: in Section 2, we provide an improved expression of the probability density function. In Section 3, we show the graph of the probability density function for the case of \( p = 3 \). The conclusion of this paper follows in Section 4.

2 Improvement of the Density Function

In this section, we improve the expression of the probability density function of the latent vector corresponding to the largest latent root of the Wishart matrix that was given originally by Sugiyama (1966), modifying his argument.

Let \( U \) be distributed as \( W_p(n, \Sigma) \). Then, it is well known that the probability density function of \( U \) is given by

\[
K\|U\|^{-\frac{p+1}{2}} \exp \left( -\frac{1}{2} \text{tr} \Sigma^{-1} U \right),
\]

where \( K = |\Sigma|^{-\frac{n}{2}} / (2\pi)^{p(n/2)} \) and \( \Gamma_p(u) = \pi^{\frac{p}{2}} \prod_{i=1}^{p} \Gamma(u - (i - 1)/2) \). Consider the following spectral decomposition:

\[
U = HD_\ell H',
\]

where \( D_\ell \) is the diagonal matrix with diagonal elements \( \ell_1 > \ell_2 > \cdots > \ell_p > 0 \) and \( H = (h_1, h_2, \ldots, h_p) \) is \( p \times p \) orthogonal matrix. Let \( R_\nu(t) \) be the single rotation matrix defined by

\[
R_\nu(t) = \begin{pmatrix}
I_{\nu-1} & 0 & 0 & 0 \\
0 & \cos t & -\sin t & 0 \\
0 & \sin t & \cos t & 0 \\
0 & 0 & 0 & I_{p-\nu-1}
\end{pmatrix},
\]

where \( I_\nu \) is the \( \nu \times \nu \) identity matrix. Then, \( H \) may be expressed as

\[
L_1(t_1) \cdots L_{p-1}(t_{p-1}) \begin{pmatrix} 1 & 0' & 0 \\ 0 & I_{p-2} & 0 \\ 0 & 0' & \varepsilon \end{pmatrix},
\]

where \( \varepsilon \) is 1 or \( -1 \), \( t_\nu = (t_{\nu, p-1}, t_{\nu, p-2}, \ldots, t_{\nu, \nu}) \) and

\[
L_\nu(t_\nu) = R_{p-1}(t_{\nu, p-1}) R_{p-2}(t_{\nu, p-2}) \cdots R_\nu(t_{\nu, \nu}).
\]
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for $0 \leq t_{ij} \leq \pi$ with $1 \leq j \leq p - 1$ and $0 \leq t_{ip - 1} < 2\pi$. In this case, we may write

$$H = H(t) = L_1(t_1) \begin{pmatrix} 1 & 0' \\ 0 & H_{p-1}(\tilde{t}) \end{pmatrix},$$

where $t_1 = (t_1, t_2, \ldots, t_{p-1})$, $\tilde{t} = (t_2, \ldots, t_{p-1})$, and $H_{p-1}(\tilde{t})$ is an orthogonal matrix of degree $p - 1$. The jacobian of the transformation $U = HDLH'$ is given by

$$J = \prod_{i<j}(d_i - d_j) \prod_{i=1}^{p-2} \prod_{j=i+1}^{p} \sin^{p-i-1}(t_{ij}). \tag{3}$$

Let $L = \text{diag}(t_2, t_3, \ldots, t_p)$. Then, we can express $\text{tr} \Sigma^{-1}U$ as

$$\text{tr} \left[ L_1(t_1)' \Sigma^{-1}L_1(t_1) \begin{pmatrix} 1 & 0' \\ 0 & H_{p-1}(\tilde{t}) \end{pmatrix} \begin{pmatrix} 1 & 0' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0' \\ 0 & H_{p-1}(\tilde{t}) \end{pmatrix} \right] + t_1 h_1' \Sigma^{-1}h_1. \tag{4}$$

From (3) and (4) the joint probability density function of $t_{ij}$ ($i = 1, \ldots, p - 1$ and $j = i, \ldots, p - 1$) and $(t_1, \ldots, t_p)$ is

$$K(\ell_1|L) \frac{c^{p^2/2}}{\pi^{p(p-1)/2}} \exp \left( -\frac{1}{2} \ell_1 h_1' \Sigma^{-1}h_1 \right) \prod_{i<j}(d_i - d_j) \prod_{i=1}^{p-2} \prod_{j=i+1}^{p} \sin^{p-i-1}(t_{ij}),$$

where $\Sigma_{p-1}(t_1)$ is $(p - 1) \times (p - 1)$ matrix obtained from deleting the first row and column of $L_1(t_1)' \Sigma^{-1}L_1(t_1)$. We use the following lemma (see e.g., Muirhead [1982]) to find the joint probability density function of $t_1 = (t_{11}, \ldots, t_{1p-1})$ and $(t_1, \ldots, t_p)$.

**Lemma 2.1** Let $S$ and $T$ be $p \times p$ positive definite matrices. Then,

$$\frac{1}{c} \int (t_1 H'SHT)^{k} \prod_{i=1}^{p-2} \prod_{j=i+1}^{p} \sin^{p-i-1}(t_{ij}) \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} dt_{ij} = \sum_{\kappa} C_{\kappa}(S)C_{\kappa}(T) C_{\kappa}(I).$$

where $\Sigma_\kappa$ stands for the sum of all possible partition $\kappa = \{k_1, \ldots, k_{p-1}\}$ of nonnegative integer $k$ satisfying $k_1 \geq \cdots \geq k_{p-1} \geq 0; C_{\kappa}(X)$ stands for the zonal polynomial corresponding to $\kappa$ and

$$c = \frac{c^{p^2/4}}{\Gamma_p(\frac{p}{2})}.$$ 

From Lemma 2.1 the joint probability density function of $t_1$ and $(t_1, \ldots, t_p)$ is given by

$$K(\ell_1|L) \frac{c^{p^2/2}}{\pi^{p(p-1)/2}} \exp \left( -\frac{1}{2} \ell_1 h_1' \Sigma^{-1}h_1 \right) \prod_{i=1}^{p-2} \prod_{j=i+1}^{p} \sin^{p-i-1}(t_{ij}) \cdot \prod_{i<j}(d_i - d_j) \sum_{\kappa} C_{\kappa}(\frac{1}{2} \Sigma_{p-1}(t_1)) C_{\kappa}(L) / k! C_{\kappa}(I_{p-1}). \tag{5}$$

Let $\ell_1 = \ell_1 x_i$ for $i = 2, \ldots, p$, the joint probability density function of $t_1$, $(x_2, \ldots, x_p)$ and $\ell_1$ is given by
where $\mathbf{L}_x = \text{diag}(x_2, \ldots, x_p)$. In order to derive the joint probability density function of $t_1$ and $\ell_1$, we must integrate (6) with respect to $(x_2, \ldots, x_p)$. The following lemma (see e.g., Muirhead [1982]) is used to integrate.

**Lemma 2.2** Let $\mathbf{A}$ be a diagonal matrix with diagonal elements $1 > \lambda_1 > \cdots > \lambda_m > 0$. Then,

$$
\int_{1 > \lambda_1 > \cdots > \lambda_m > 0} |\mathbf{A}|^{\frac{1 - \nu}{2}} \prod_{k=1}^m (1 - \lambda_k)^{-\nu} \prod_{i < j} \lambda_i \lambda_j C_{\nu}(\mathbf{A}) \prod_{i=1}^m d\lambda_i
= \frac{\Gamma_m(\frac{m}{2})}{\pi^{\frac{m(m-1)}{2}}} \frac{(t)_k \Gamma_m(t) \Gamma_m(u)}{(t + u)_k \Gamma_m(t + u)} C_{\nu}(\mathbf{I}_{p-1}).
$$

where

$$(a)_k = \prod_{i=1}^k (a - \frac{1}{2}(i - 1))h_i, \quad (b)_k = \prod_{j=1}^k (b + (j - 1)).$$

Integrating (6) with respect to $(x_2, \ldots, x_p)$ the joint probability density function of $t_1$ and $\ell_1$ is given by

$$
\frac{|\mathbf{Σ}|^{\frac{1}{2}}}{2^{\frac{p-1}{2}} \Gamma_p(\frac{1}{2})} \prod_{j=1}^{p-1} \frac{\Gamma_p((\frac{p+1}{2})(\frac{j}{2}) \frac{1}{2})}{\Gamma_p(\frac{1}{2})} \frac{1}{\Gamma_{p-1}(\frac{1}{2})} \frac{1}{\Gamma_{p-1}(\frac{1}{2})} \exp \left( -\frac{1}{2} \ell_1 h_1^\top \mathbf{Σ}^{-1} h_1 \right)
\cdot \sum_{\kappa=0}^{\infty} \sum_{\ell_1}^{\infty} \frac{(\frac{1}{2})_{\ell_1}}{\ell_1!} \frac{C_{\kappa}(-\frac{1}{2} \mathbf{Σ}_{p-1}(t_1))}{\kappa!} \prod_{j=1}^{p-2} \sin^{\ell_1 - 2} t_{1j}
= \frac{|\mathbf{Σ}|^{\frac{1}{2}}}{2^{\frac{p-1}{2}} \Gamma_p(\frac{1}{2})} \prod_{j=1}^{p-1} \frac{\Gamma_p((\frac{p+1}{2})(\frac{j}{2}) \frac{1}{2})}{\Gamma_p(\frac{1}{2})} \frac{1}{\Gamma_{p-1}(\frac{1}{2})} \frac{1}{\Gamma_{p-1}(\frac{1}{2})} \exp \left( -\frac{1}{2} \ell_1 h_1^\top \mathbf{Σ}^{-1} h_1 \right)
\cdot 1_{\mathbf{F}_1}(\frac{1}{2}, \frac{n+1}{2}; -\frac{1}{2} \ell_1 \mathbf{Σ}_{p-1}(t_1)) \prod_{j=1}^{p-2} \sin^{\ell_1 - 2} t_{1j},
$$

where $1_{\mathbf{F}_1}(a; c; \mathbf{X})$ denotes the hypergeometric function with the matrix argument defined by

$$
1_{\mathbf{F}_1}(a; c; \mathbf{X}) = \sum_{k=0}^{\infty} \sum_{c} \frac{(a)_k C_{\kappa}(\mathbf{X})}{k!}.
$$

These calculations are the same as those of Sugiyama (1966). The positive term series of expression of the joint probability density function is obtained by the Kummer transformation (see e.g., Muirhead [1982]):

$$
1_{\mathbf{F}_1}(a; c; \mathbf{X}) = e^{\text{tr}(\mathbf{X})} \cdot 1_{\mathbf{F}_1}(c - a; c; -\mathbf{X}).
$$
Lemma 2.3 (Poincaré separation theorem) The following lemma is needed to show that the joint probability density function in (7) can be expressed by

\[
\frac{(\Sigma^{-1})^{\frac{1}{2}}}{2\pi^{p/2} \Gamma_p(\frac{p}{2})} \frac{\Gamma_{p-1}(\frac{p+2}{2})}{\Gamma_{p-1}(\frac{p}{2}+1)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{p+2}{2})_{\kappa}}{(\frac{p+1}{2})_{\kappa}} \frac{C_{n}(\lambda \Sigma^{-1}(t_1))}{k!} C_{n}^{\kappa}(\lambda \Sigma^{-1}(t_1)) \exp\left(-\frac{1}{2} \sum_{j=1}^{p} \sin^2 t_{1j}\right) \prod_{j=1}^{p} \sin^{p-2} t_{1j}.
\]

Integrating with respect to \( t_1 \), the joint probability density function of \( t_1 \) is given by

\[
\frac{(\Sigma^{-1})^{\frac{1}{2}}}{2\pi^{p/2} \Gamma_p(\frac{p}{2})} \frac{\Gamma_{p-1}(\frac{p+2}{2})}{\Gamma_{p-1}(\frac{p}{2}+1)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{p+2}{2})_{\kappa}}{(\frac{p+1}{2})_{\kappa}} C_{n}(\lambda \Sigma^{-1}(t_1)) \exp\left(-\frac{1}{2} \sum_{j=1}^{p} \sin^2 t_{1j}\right) \prod_{j=1}^{p} \sin^{p-2} t_{1j},
\]

where \( B_p(\alpha, \beta) = \Gamma_p(\alpha) \Gamma_p(\beta) / \Gamma_p(\alpha+\beta) \). We must confirm the uniform convergence of the positive series. The following lemma is needed to show that

**Lemma 2.3** (Poincaré separation theorem) Let \( A \) be an \( p \times p \) symmetric matrix and \( B \) be an \( p \times h \) matrix satisfying \( B'B = I_h \). Then, for \( i = 1, \ldots, h \), it follows that:

\[
\lambda_{p-i+i}(A) \leq \lambda_i(B'AB) \leq \lambda_i(A),
\]

where \( \lambda_j(A) \) denotes the \( j \)-th largest latent root of \( A \).

Let \( \Sigma = \text{diag}(1/\lambda_1, \ldots, 1/\lambda_2) \), where \( \lambda_j \) denotes the \( j \)-th largest latent root of \( \Sigma \). Using Lemma 2.3,

\[
C_{n}(\Sigma^{-1}(t_1)) \leq C_{n}^{\kappa}(\Sigma),
\]

for any \( t_1 \) and thus

\[
\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{p+2}{2})_{\kappa}}{(\frac{p+1}{2})_{\kappa}} C_{n}(\lambda \Sigma^{-1}(t_1)) \leq \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{p+2}{2})_{\kappa}}{(\frac{p+1}{2})_{\kappa}} C_{n}^{\kappa}(\lambda \Sigma^{-1}(t_1)).
\]

which is bounded by \( 1 F_2(np/2; \text{tr} \Sigma; \text{tr} \Sigma^{-1}) \). Noting that \( \text{tr} \Sigma / \text{tr} \Sigma^{-1} \) is less than 1, infinite series in (8) is uniformly convergent in the wider sense.

**Theorem 2.1** The joint probability density function of \( t_{11}, \ldots, t_{1p} \) is given by (8).

When \( p = 2 \), it can be verified after some calculations that (8) is identical to the probability density function given in Sugiyama (1965), stating that:

\[
\frac{1}{\pi(n+1)} \left\{ 4 |\Sigma| \right\}^{\frac{1}{2}} \binom{n+1}{2} \sum_{t} F_1 \left(1, n; \frac{n+1}{2}; x(t)\right) - \frac{3}{2} x F_1 \left(1, n; \frac{n+3}{2}; x(t)\right),
\]

where \( x(t) = (\lambda_1 \cos^2 t + \lambda_2 \sin^2 t) / \text{tr} \Sigma \).
3 Numerical Result

In this section we drew the graphs of the joint probability density function of \( t_1 \) when \( p = 3 \). From (8), the joint probability density function of \( t_{11} \) and \( t_{12} \) is given by

\[
\frac{1}{\Gamma_3(f/(1-f))} \frac{\gamma((\frac{f}{2})n)}{\Gamma(\frac{f}{2})} B_2 \left( \frac{5}{2}, \frac{n-1}{2} \right) \cdot \sum_{k=0}^{\infty} \frac{\gamma((\frac{f}{2})n)}{\Gamma(\frac{f}{2})} C_n \left( \frac{1}{\Gamma_2^2 \Sigma} \right) \sin t_{11}. \tag{10}
\]

From James (1968), it is known that for any \( 2 \times 2 \) positive definite matrix \( A \),

\[
C_n(A) = C_n(I_2) \frac{1}{(a_1 a_2) \xi^2 P_{-k_1-k_2} \left( \frac{1}{2}(a_1 + a_2)(a_1 a_2)^{-\frac{1}{2}} \right)},
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix, \( a_1 \) and \( a_2 \) are the ordered latent roots of \( A \), and \( P_{-k_1-k_2}(u) \) is the Legendre polynomials defined by

\[
P_{-k_1-k_2}(u) = (-1)^k(2q)!/(2^{2q}(q!)^2)^{-1} \cdot 2F_1 \left( -q, q + 1; 2; u^2 \right)
\]

for \( k_1 - k_2 = 2q \) and

\[
P_{-k_1-k_2+1}(u) = (-1)^k(2q + 1)!/(2^{2q}(q!)^2)^{-1} \cdot 2F_1 \left( -q, q + \frac{3}{2}; 2; u^2 \right)
\]

for \( k_1 - k_2 = 2q + 1 \). It is known that

\[
C_n(I_2) = 2^{2k} k! (1)_{\xi^2} \frac{\prod_{i=1}^{m} (2k_i - 2k_i - i + j)}{\prod_{i=1}^{m} (2k_i + m - i)!},
\]

where \( m \) denotes the number of non-zero parts of \( \kappa \) (see e.g.,Muirhead[1982]). We can construct a spectrum decomposition, whereby: \( \Sigma = \Gamma \Lambda \Gamma' \), where \( \Gamma \) is a \( 3 \times 3 \) orthogonal matrix and \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \). From (2), we can write

\[
\Gamma = (\gamma_1, \gamma_2, \gamma_3)
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{12} & -\sin \theta_{12} \\
0 & \sin \theta_{12} & \cos \theta_{12}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{11} & -\sin \theta_{11} & 0 \\
\sin \theta_{11} & \cos \theta_{11} & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{22} & -\sin \theta_{22} \\
0 & \sin \theta_{22} & \cos \theta_{22}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]

where \( 0 \leq \theta_{11}, \theta_{12} \leq \pi, \ 0 \leq \theta_{22} < 2\pi \). It follows that

\[
\gamma_1 = \begin{pmatrix}
\cos \theta_{11}, & \sin \theta_{11} \cos \theta_{12}, & \sin \theta_{11} \sin \theta_{12}
\end{pmatrix}.
\tag{11}
\]

We can choose \( \theta_{11}, \theta_{12} \) to characterize \( \gamma_1 \). In this paper, we drew the density function for the following
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cases (A to D). Multiple root ($\Lambda = \text{diag}(3,2,2)$):

Case $A_1$: $\gamma_1 = (0,0,1)'$,
Case $B_1$: $\gamma_1 = \left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right)'$,
Case $C_1$: $\gamma_1 = \left(1/\sqrt{2}, 0, 1/\sqrt{2}\right)'$,
Case $D_1$: $\gamma_1 = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)'$;

Simple root ($\Lambda = \text{diag}(3,2,1)$):

Case $A_2$: $\gamma_1 = (0,0,1)'$,
Case $B_2$: $\gamma_1 = \left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right)'$,
Case $C_2$: $\gamma_1 = \left(1/\sqrt{2}, 0, 1/\sqrt{2}\right)'$,
Case $D_2$: $\gamma_1 = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)'$;

Case $A_1$ represents the transformation of the first element of the original variable to the first principal component. Cases $B_1$ and $C_1$ represent the subtraction and sum of the first and third variables, respectively. Case $D_1$ represents the total variation in the matrix. Cases $A_2$, $B_2$, $C_2$, and $D_2$ represent the same transformations to principal components and variability as in Cases $A_1$, $B_1$, $C_1$, $D_1$, respectively, when $\Sigma$ is simple. We drew the graphs when $p = 3$ and $n = 4, 10, 50$. 
3.1 Multiple roots

We checked the numerical convergence of the infinite series of (8). Table 1 shows that the sum up to \( k = 70 \) is sufficient to obtain 3 digits of accuracy for \( n = 4, k = 200 \) for \( n = 10 \), and \( k = 210 \) for \( n = 50 \).

3.1.1 Case A: \( \gamma_1 = (0, 0, 1)' \)

The probability density function in (8) is shown in Figures 1-3 when \( \gamma_1 = (1, 0, 0) \) for \( n = 4, 10, 50 \). The graphs are symmetric with respect to \((\pi/2, \pi/2)\).

3.1.2 Case B: \( \gamma_1 = ( -1/\sqrt{2}, 0, 1/\sqrt{2})' \)

The probability density function in (8) is shown in Figures 4-6 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (3\pi/4, \pi/2, \pi/4)\) for \( n = 4, 10, 50 \).

3.1.3 Case C: \( \gamma_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})' \)

The probability density function in (8) is shown in Figures 7-9 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (\pi/4, \pi/2, \pi/4)\) for \( n = 4, 10, 50 \).

3.1.4 Case D: \( \gamma_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})' \)

The probability density function in (8) is shown in Figures 10-12 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (\arccos(1/\sqrt{3}), \pi/4, \pi/4)\) for \( n = 4, 10, 50 \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \gamma_1 )</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( (0, 0, 1)' )</td>
<td>Figures 1-3 when ( \gamma_1 = (1, 0, 0) ) for ( n = 4, 10, 50 )</td>
</tr>
<tr>
<td>B</td>
<td>( (-1/\sqrt{2}, 0, 1/\sqrt{2})' )</td>
<td>Figures 4-6 when ((\theta_{11}, \theta_{12}, \theta_{22}) = (3\pi/4, \pi/2, \pi/4)) for ( n = 4, 10, 50 )</td>
</tr>
<tr>
<td>C</td>
<td>( (1/\sqrt{2}, 0, 1/\sqrt{2})' )</td>
<td>Figures 7-9 when ((\theta_{11}, \theta_{12}, \theta_{22}) = (\pi/4, \pi/2, \pi/4)) for ( n = 4, 10, 50 )</td>
</tr>
<tr>
<td>D</td>
<td>( (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})' )</td>
<td>Figures 10-12 when ((\theta_{11}, \theta_{12}, \theta_{22}) = (\arccos(1/\sqrt{3}), \pi/4, \pi/4)) for ( n = 4, 10, 50 )</td>
</tr>
</tbody>
</table>

Table 1: Right-hand side of (9) for \( n = 4, 10, 50 \), \( p = 3 \) and \( \Lambda = \text{diag}(3, 2, 2) \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \gamma_1 )</th>
<th>Right-hand side of (9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( (0, 0, 1)' )</td>
<td>( n = 4 ), ( n = 10 ), ( n = 50 )</td>
</tr>
<tr>
<td>B</td>
<td>( (-1/\sqrt{2}, 0, 1/\sqrt{2})' )</td>
<td>( n = 4 ), ( n = 10 ), ( n = 50 )</td>
</tr>
<tr>
<td>C</td>
<td>( (1/\sqrt{2}, 0, 1/\sqrt{2})' )</td>
<td>( n = 4 ), ( n = 10 ), ( n = 50 )</td>
</tr>
<tr>
<td>D</td>
<td>( (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})' )</td>
<td>( n = 4 ), ( n = 10 ), ( n = 50 )</td>
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</tbody>
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<table>
<thead>
<tr>
<th>( k )</th>
<th>( n = 4 )</th>
<th>( n = 10 )</th>
<th>( n = 50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 10</td>
<td>73.1265</td>
<td>398.2075</td>
<td>3144.8170</td>
</tr>
<tr>
<td>10 - 20</td>
<td>35.3206</td>
<td>638.4612</td>
<td>34827.9060</td>
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<td>20 - 30</td>
<td>5.5627</td>
<td>254.3336</td>
<td>93050.2335</td>
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<td>30 - 40</td>
<td>0.6156</td>
<td>57.4889</td>
<td>124883.2652</td>
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<td>40 - 50</td>
<td>0.0574</td>
<td>9.5579</td>
<td>109915.2269</td>
</tr>
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<td>150 - 160</td>
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<td>1.82878</td>
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<tr>
<td>160 - 170</td>
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<td>0.0000</td>
<td>0.519897</td>
</tr>
<tr>
<td>170 - 180</td>
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<td>0.0000</td>
<td>0.182878</td>
</tr>
<tr>
<td>180 - 190</td>
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<td>0.0000</td>
<td>0.0519897</td>
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<td>0.0000</td>
<td>0.0182878</td>
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<td>200 - 210</td>
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<td>Total</td>
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<td>1359.5323</td>
<td>498236.7287</td>
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</table>
Behavior of Latent Vector of Trivariate Wishart Matrix

- Case A: \( \gamma_1 = (0, 0, 1)' \)

- Case B: \( \gamma_1 = (-1/\sqrt{2}, 0, 1/\sqrt{2})' \)

- Case C: \( \gamma_1 = (1/\sqrt{3}, 0, 1/\sqrt{3})' \)

- Case D: \( \gamma_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})' \)
3.2 Simple root

We checked the numerical convergence of the infinite series of (8) when \(\Lambda = \text{diag}(3, 2, 1)\). Table 2 shows that the sum up to \(k = 110\) is sufficient to obtain 3 digits of accuracy for \(n = 4\), \(k = 170\) for \(n = 10\), and \(k = 370\) for \(n = 50\).

3.2.1 Case A: \(\gamma_1 = (0, 0, 1)^t\)

We studied the case in which Figures 13-15 show the probability density function (8) when \(\Sigma = \text{diag}(3, 2, 1)\) for \(n = 4\), \(10\), \(50\). The graph shows that large amount of probability mass is concentrated on the line \(t_{11} = \pi/2\).

3.2.2 Case B: \(\gamma_1 = (\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^t\)

The probability density function in (8) is shown in Figures 16-18 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (3\pi/4, \pi/2, \pi/4)\) for \(n = 4\), \(10\), \(50\). These figures show that the curve of the points of the highly concentrated probability density is not linear. This is the case in all subsequent Figures.

3.2.3 Case C: \(\gamma_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^t\)

The probability density function in (8) is shown in Figures 19-21 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (\frac{\pi}{4}, \pi/2, \pi/4)\) for \(n = 4\), \(10\), \(50\).

3.2.4 Case D: \(\gamma_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^t\)

The probability density function in (8) is shown in Figures 22-24 when \((\theta_{11}, \theta_{12}, \theta_{22}) = (\arccos(1/\sqrt{3}), \pi/4, \pi/4)\) for \(n = 4\), \(10\), \(50\).

Table 2: Right-hand side of (9) for \(n = 4, 10, 50\), \(p = 3\) and \(\Lambda = \text{diag}(3, 2, 1)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n = 4)</th>
<th>(n = 10)</th>
<th>(n = 50)</th>
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<td>894.7891</td>
<td>9866.7591</td>
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<td>341.1446</td>
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<td>20 ~ 30</td>
<td>52.6759</td>
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<td>30 ~ 40</td>
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<td>60 ~ 70</td>
<td>0.0366</td>
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<td>160 ~ 170</td>
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<td>:</td>
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<tr>
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<tr>
<td>360 ~ 370</td>
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<td>344.9133</td>
<td>10519.1656</td>
<td>904210197.6290</td>
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</tbody>
</table>
Behavior of Latent Vector of Trivariate Wishart Matrix

- Case A:
  \[ \gamma_1 = (0, 0, 1) \]

  Figure 13: \( n = 4 \)  
  Figure 14: \( n = 10 \)  
  Figure 15: \( n = 50 \)

- Case B:
  \[ \gamma_1 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}) \]

  Figure 16: \( n = 4 \)  
  Figure 17: \( n = 10 \)  
  Figure 18: \( n = 50 \)

- Case C:
  \[ \gamma_1 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}) \]

  Figure 19: \( n = 4 \)  
  Figure 20: \( n = 10 \)  
  Figure 21: \( n = 50 \)

- Case D:
  \[ \gamma_1 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}) \]

  Figure 22: \( n = 4 \)  
  Figure 23: \( n = 10 \)  
  Figure 24: \( n = 50 \)
4 Conclusion

In this paper, we have derived an improved expression of the probability density function of the latent vector corresponding to the largest latent root of Wishart matrix given originally in Sugiyama (1966). To illustrate our improvement, we drew the graph of the probability density function of the polar coordinates of the latent vectors when \( p = 3 \) and confirmed that the concentration of the distribution of the latent vector increases as \( n \) increases. As a future study, we will provide the exact confidence region of the vector. This study is currently underway in our laboratory.

Acknowledgment

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References